

The cascade of orthogonal roots and the coadjoint structure of the nilradical of a Borel subgroup of a semisimple Lie group

To the memory of a dear friend, I. M. Gelfand, one of the great mathematicians of the 20th century

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Abstract: Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g} = \text{Lie } G$. Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and let H, N, B be Lie subgroups of G corresponding respectively to $\mathfrak{h}, \mathfrak{n}$ and \mathfrak{b} . We may identify \mathfrak{n}_- with the dual space to \mathfrak{n} . The coadjoint action of N on \mathfrak{n}_- extends to an action of B on \mathfrak{n}_- . There exists a unique nonempty Zariski open orbit X of B on \mathfrak{n}_- . Any N -orbit in X is a maximal coadjoint orbit of N in \mathfrak{n}_- . The cascade of orthogonal roots defines a cross-section \mathfrak{r}_-^\times of the set of such orbits leading to a decomposition

$$X = N/R \times \mathfrak{r}_-^\times.$$

This decomposition, among other things, establishes the structure of $S(\mathfrak{n})^n$ as a polynomial ring generated by the prime polynomials of H -weight vectors in $S(\mathfrak{n})^n$. It also leads to the multiplicity 1 of H weights in $S(\mathfrak{n})^n$.

Key words: cascade of orthogonal roots, Borel subgroups, nilpotent coadjoint action.

MSC (2010) subject codes: representation theory, invariant theory.

0. Introduction

0.1. Let \mathfrak{g} be a complex semisimple Lie algebra and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . The Killing form (x, y) , denoted by \mathcal{K} , on \mathfrak{g} induces a nonsingular bilinear form (μ, ν) on the dual space \mathfrak{h}^* to \mathfrak{h} . Let $\Delta \subset \mathfrak{h}^*$ be the set of roots

corresponding to $(\mathfrak{h}, \mathfrak{g})$. For each $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. Let \mathfrak{b} be a Borel subalgebra of \mathfrak{g} which contains \mathfrak{h} and let \mathfrak{n} be the nilradical of \mathfrak{b} . Let

$$\mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$$

be a triangular decomposition of \mathfrak{g} . Then a choice Δ_+ (respectively Δ_-) of positive (resp. negative) roots is chosen so that $\Delta_+ = \{\varphi \mid e_\varphi \in \mathfrak{n}\}$ (resp. $\{\varphi \mid e_{-\varphi} \in \mathfrak{n}_-\}$). One has $\Delta_- = -\Delta_+$. If $\mathfrak{s} \subset \mathfrak{g}$ is a Lie subalgebra, then $S(\mathfrak{s})$ and $U(\mathfrak{s})$ will denote respectively the symmetric and enveloping algebras of \mathfrak{s} . We are mainly concerned with the case where $\mathfrak{s} = \mathfrak{n}$ and with the structure of the space of \mathfrak{n} -invariants $S(\mathfrak{n})^{\mathfrak{n}}$ (or equivalently $\text{cent } U(\mathfrak{n})$) and with the action of \mathfrak{h} on $S(\mathfrak{n})^{\mathfrak{n}}$ and $\text{cent } U(\mathfrak{n})$.

Let G be a Lie group such that $\mathfrak{g} = \text{Lie } G$ and let H , N , and B be the Lie subgroups which correspond respectively to $\mathfrak{h}, \mathfrak{n}$ and \mathfrak{b} . Using \mathcal{K} we may identify \mathfrak{n}_- as the dual space to \mathfrak{n} so that the coadjoint action of N defines an action of N on \mathfrak{n}_- . Furthermore since B normalizes N there is accompanied an action of B (and in a particular H) on \mathfrak{n}_- . Many of the results in this paper are quite old and were cited in [J]. The techniques in this paper are algebra-geometric in nature and are quite different from those in [J]. We will see that B has a unique Zariski open and (dense) orbit X in \mathfrak{n}_- . The results arise from the rather elegant structure of the coadjoint action of N on the affine variety X and from the equally elegant action of H on these N -coadjoint orbits. The main tool is the use of our well-known cascade of orthogonal roots. The cascade plays a major role in a number of papers. In particular the cascade has been used in [J] and [L-W]. We will recall the definition of the cascade \mathcal{B} in §1 below and elaborate on some of its properties. Among other things \mathcal{B} is a special maximal set of strongly orthogonal roots. Let $m = \text{card } \mathcal{B}$ and we will write $\mathcal{B} = \{\beta_1, \dots, \beta_m\}$.

Let $\mathfrak{r}_- \subset \mathfrak{n}_-$ be the span of $\{e_{-\beta_i}\}$, $i = 1, \dots, m$, and let \mathfrak{r}_-^\times be the Zariski open subset of \mathfrak{r}_- defined by the condition that if $e \in \mathfrak{r}_-$, then $e \in \mathfrak{r}_-^\times$ if all the

coefficients of e relative to the $e_{-\beta_i}$ are nonzero. Let T be the toroidal subgroup of H defined so that $\beta_i^\vee, i = 1, \dots, m$, is a basis of $\text{Lie } T$. Then T operates on \mathfrak{t}_- and \mathfrak{t}_-^\times is the unique Zariski open orbit of T in \mathfrak{t}_- . It is clear that T operates simply and transitively on \mathfrak{t}_-^\times so that as affine varieties $T \cong \mathfrak{t}_-^\times$. In this paper we will prove **(1)** that $\mathfrak{t}_-^\times \subset X$ and **(2)** \mathfrak{t}_-^\times is a cross-section of the set of N -coadjoint orbits in X . In addition, **(3)** every N orbit on X is a maximal coadjoint orbit of N in \mathfrak{n}_- and furthermore **(4)**, the isotropy group R of N at any point p in \mathfrak{t}_-^\times is independent of p . Moreover if $A(Y)$ denotes the affine ring of an affine variety Y , then **(5)** one has, as affine varieties, $X \cong N/R \times \mathfrak{n}_-^\times$ so that

$$A(X) \cong A(N/R) \otimes A(T) \quad (0.1)$$

and **(6)**

$$A(X)^N \cong A(T), \quad (0.2)$$

noting **(7)**, that $A(T)$ is the character ring of the torus T and is generated by the cascade \mathcal{B} which, in fact, is a basis of the character ring.

Returning to the description of $S(\mathfrak{n})^\mathfrak{n}$ we establish that **(a)** $S(\mathfrak{n})^\mathfrak{n}$ embeds naturally in $A(T)$ and that $S(\mathfrak{n})^\mathfrak{n}$ and $A(T)$ have the same quotient field. Let \widehat{T} be the character group of T so that $\widehat{T} \subset A(T)$. For any $\xi \in \widehat{T}$ let $\nu(\xi)$ be the corresponding H -weight so that

$$\{\nu(\xi) \mid \xi \in \widehat{T}\} \text{ is the free abelian group generated by } \mathcal{B}.$$

Let $Q = S(\mathfrak{n})^\mathfrak{n} \cap \widehat{T}$. One readily has that $\nu(\xi)$ is dominant for any $\xi \in Q$. Now let P be the set of all $\xi \in Q$ which, as polynomials on \mathfrak{n}_- , are prime. We prove **(b)** $\text{card } P = m$ and **(c)** if $P = \{\xi_1, \dots, \xi_m\}$, then for $i = 1, \dots, m$, the ξ_i are algebraically independent and $\nu_i = \nu(\xi_i)$ are linearly independent. Furthermore if $d = (d_1, \dots, d_m) \in \mathbb{Z}^m$ and

$$\xi^d = \xi_1^{d_1} \cdots \xi_m^{d_m}, \quad (0.3)$$

note that

$$\nu(\xi^d) = \sum_{i=1}^m d_i \nu_i. \quad (0.4)$$

We prove **(d)**

$$\widehat{T} = \{\xi^d \mid d \in \mathbb{Z}^m\}, \quad (0.5)$$

and if $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers, then we also establish **(e)**

$$Q = \{\xi^d \mid d \in \mathbb{N}^m\} \quad (0.6)$$

so that **(f)**

$$S(\mathfrak{n})^{\mathfrak{n}} \text{ is the polynomial ring } \mathbb{C}[\xi_1, \dots, \xi_m] \text{ and } Q \text{ is a weight basis of this ring.} \quad (0.7)$$

In particular **(g)** every weight in Q has multiplicity 1 and if $\eta \in Q$, then **(h)** writing

$$\eta = \xi^d, \text{ where } d \in \mathbb{N}^m, \text{ is the prime decomposition of the polynomial } \eta. \quad (0.8)$$

Furthermore since any $\nu(\eta)$ is dominant for any $\eta \in Q$ the coefficients k_j in the expansion $\nu(\eta) = \sum_{j=1}^m k_j \beta_j$ are nonnegative integers and **(i)** as a polynomial on \mathfrak{n}_- ,

$$\deg \eta = \sum_{j=1}^m k_j. \quad (0.9)$$

Remark 0.1. Given a dominant weight, we constructed in [K] (modifying the method of Lipsman–Wolf) an element $f_{(k)} \in S(\mathfrak{n})^{\mathfrak{n}}$ of degree k . If the dominant weight is ρ , equal to one-half the sum of the positive roots, then $\nu(f_{(k)}) = 2\rho$, and one readily shows that all the elements ξ_i in P may be given as the prime factors of $f_{(k)}$.

Finally returning to (0.1) one establishes **(j)** that $S(n)^{\mathfrak{n}}$ “separates” all the maximal N -coadjoint orbits that lie in X .

Remark 0.2. A number of results in [J] were unknown to us when [J] was written. For example we were unaware that the set

$$\{\nu(\eta) \mid \eta \in Q\}$$

included all the dominant elements in the lattice L generated by \mathcal{B} . This, however, follows from an argument in [K] which proves that if $\eta \in L$ and $\nu(\eta)$ is dominant, then $\eta^4 \in Q$, which by (0.6), implies that $\eta \in Q$. Joseph in [J] does not deal with prime polynomials. Instead he sets up a bijection of \mathcal{B} with certain generators of $S(\mathfrak{n})^{\mathfrak{n}}$,

$$\beta_j \mapsto \eta_j. \quad (0.10)$$

Furthermore he very cleverly determines, by a sort of Gram-Schmidt process, the $m \times m$ matrix s_{ij} where

$$\nu(\eta_j) = \sum_{i=1}^m s_{ij} \beta_i \quad (0.11)$$

In addition very useful information is given in his tables II and III. Among other things the $\nu(\eta_j)$ are expressed in terms of the fundamental representations of \mathfrak{g} .

0.2. The results in this paper were inspired by Dixmier's result [D] for the special case where $G = Sl(n, \mathbb{C})$.

1. The cascade of orthogonal roots

1.1. Let $\ell = \text{rank } \mathfrak{g}$ and let $\Pi \subset \Delta_+$ be the set of simple positive roots. For each $\varphi \in \Delta_+$ let $n_\alpha(\varphi) \in \mathbb{Z}_+$ be such that $\varphi = \sum_{\alpha \in \Pi} n_\alpha(\varphi) \alpha$. Now let $\Pi(\varphi) = \{\alpha \in \Pi \mid n_\alpha(\varphi) > 0\}$. Then, as one knows, $\Pi(\varphi)$ is a connected subset of Π . Hence there is a unique complex simple Lie subalgebra $\mathfrak{g}(\varphi)$ of \mathfrak{g} , with Cartan subalgebra \mathfrak{h} , having $\Pi(\varphi)$ as a set of simple positive roots. Let $\Delta(\varphi) \subset \Delta$ be the set of roots of $(\mathfrak{h}, \mathfrak{g}(\varphi))$ and $\Delta(\varphi)_+ = \Delta(\varphi) \cap \Delta_+$. Let $\mathfrak{b}(\varphi) = \mathfrak{b} \cap \mathfrak{g}(\varphi)$ and $\mathfrak{n}(\varphi) = \mathfrak{n} \cap \mathfrak{g}(\varphi)$.

Let $\beta \in \Delta_+$. We will say that β is locally high if β is the highest root of $\mathfrak{g}(\beta)$. If $\beta \in \Delta_+$ is locally high, let $E(\beta) = \{\varphi \in \Delta(\beta) \mid (\varphi, \beta) > 0\}$ and let $\mathfrak{e}(\beta)$ be the span of e_φ for $\varphi \in E(\beta)$. Let $h^\vee(\beta)$ be the dual Coxeter number of $\mathfrak{g}(\beta)$. Then, regarding the one-dimensional Lie algebra as a Heisenberg Lie algebra, one knows

Proposition 1.1.

- (1) $\mathfrak{e}(\beta) \subset \mathfrak{n}(\beta)$ and $\mathfrak{e}(\beta)$ is an ideal in $\mathfrak{b}(\beta)$
- (2) $\mathfrak{e}(\beta)$ is a Heisenberg Lie algebra of dimension $2h^\vee(\beta) - 3$ and with center $\mathbb{C}e_\beta$
- (3) $2(\beta, \varphi)/(\beta, \beta) = 1, \forall \varphi \in E(\beta)/\{\beta\}$.

For any $\varphi \in \Delta$ let $\mathfrak{u}(\varphi)$ be the TDS spanned by h_φ, e_φ and $e_{-\varphi}$ and let $\mathfrak{g}(\beta)^\circ$ be the semisimple Lie subalgebra (possibly zero) of $\mathfrak{g}(\beta)$ spanned by all $\mathfrak{u}(\varphi)$ where $\varphi \in \Delta(\beta)$ and $(\beta, \varphi) = 0$.

Remark 1.2. One notes that the highest root of any simple component of $\mathfrak{g}(\beta)^\circ$ is locally high. It is necessarily orthogonal to β .

Introduce the usual partial ordering in the weight lattice where

$$\nu > \mu \tag{1.1}$$

if $\nu - \mu$ is a sum of positive roots.

A sequence

$$C = \{\beta_1, \dots, \beta_k\} \tag{1.2}$$

of positive roots will be called a chain cascade if β_1 is the highest root of a simple component of \mathfrak{g} and, inductively, if $1 < j \leq k$, and β_i has been given for $1 \leq i < j$ and is locally high for all such i then β_j is the highest root of a simple component of $\mathfrak{g}(\beta_{j-1})^\circ$.

Remark 1.3. One notes a chain cascade is simply ordered. For the chain cascade above β_1 is the maximal element and β_k is the minimal element.

Let \mathcal{B} be the set (cascade) of all positive roots β which are members of some chain cascade. As an immediate consequence of Remark 1.2 one has

Proposition 1.4. *Any root β in \mathcal{B} is locally high.*

Clearly for any positive root φ one uniquely defines a chain cascade

$$C(\varphi) = \{\beta_1, \dots, \beta_k\} \quad (1.3)$$

by the condition that (1) $\varphi \in \Delta(\beta_i)_+$, $i = 1, \dots, k$, and (2) $(\varphi, \beta_i) = 0$, $i = 1, \dots, k-1$, and $(\varphi, \beta_k) > 0$. In the notation of (1.2) clearly

$$C = C(\beta_k).$$

One readily also notes

Proposition 1.5. *For $\varphi, \varphi' \in \Delta_+$ one has*

$$C(\varphi) = C(\varphi') \iff \text{there exists } \beta \in \mathcal{B} \text{ such that } \varphi, \varphi' \in E(\beta) \quad (1.4)$$

in which case β is the minimal element of $C(\varphi) = C(\varphi')$. In particular one has the disjoint union

$$\Delta_+ = \cup_{\beta \in \mathcal{B}} E(\beta) \quad (1.5)$$

and the consequential direct sum (as a vector space) of Heisenbergs

$$\mathfrak{n} = \oplus_{\beta \in \mathcal{B}} \mathfrak{e}(\beta). \quad (1.6)$$

Lemma 1.6. *Any two distinct elements of a chain cascade C are strongly orthogonal.*

Proof. By definition it is clear that any distinct members of C are orthogonal. Without loss assume C is given by (1.2) and $1 \leq i < j \leq k$. Then $\beta_j \in \Delta(\beta_i)$. But then $\beta_i + \beta_j$ cannot be a root since β_i is the highest root of $\mathfrak{g}(\beta_i)$. QED

Given a chain cascade C , say given by (1.2), it is clear that $C' = \{\beta_1, \dots, \beta_j\}$ is a chain cascade for any $j \leq k$. Given two chain cascades C and C' we will say that C' is a subchain of C if C and C' are of this form. Also subsets Δ^1 and Δ^2 of Δ will be called totally disjoint if any element of Δ_1 is strongly orthogonal to every element of Δ_2 .

Proposition 1.7. *Let $\beta, \beta' \in \mathcal{B}$ be distinct. If $C(\beta)$ is not a subchain of $C(\beta')$ and vice versa, then not only are β and β' strongly orthogonal but in fact $\Delta(\beta)$ and $\Delta(\beta')$ are totally disjoint.*

Proof. Without loss assume that $\beta = \beta_k$ in the notation of (1.2) so that in that notation $C(\beta) = C$. Let $C(\beta') = \{\beta'_1, \dots, \beta'_{k'}\}$. Without loss we may assume that $k \leq k'$. By our assumption on subchains one has $\beta'_k \neq \beta_k$. Let $j \leq k$ be minimal such that $\beta'_j \neq \beta_j$. If $j = 1$, then the result is clear since e_β and $\tilde{e}_{\beta'}$ lie in different simple components of g . Assume $j > 1$. Then $\beta_{j-1} = \beta'_{j-1}$, but then e_β and $e_{\beta'}$ lie in different simple components of $\mathfrak{g}(\beta_{j-1})^\circ$. The result then follows. QED

Theorem 1.8. *The set \mathcal{B} is a maximal set of strongly orthogonal roots.*

Proof. That \mathcal{B} is a set of strongly orthogonal roots follows from Lemma 1.6 and Proposition 1.7. That it is maximal follows from the disjoint union (1.5). QED

We call \mathcal{B} the cascade of strongly orthogonal roots.

1.4. Let W be the Weyl group of \mathfrak{g} operating in \mathfrak{h} and \mathfrak{h}^* . For $\beta \in \mathcal{B}$ let $W(\beta) \subset W$ be the Weyl group of $\mathfrak{g}(\beta)$. Reluctantly submitting to common usage, let w_o be the long element of W and let $w_o(\beta)$ be the long element of $W(\beta)$. For any $\beta \in \mathcal{B}$ let $s_\beta \in W$ be the reflection defined by β .

Proposition 1.9. *Let $\beta, \beta' \in \mathcal{B}$. Then $\Delta(\beta)$ is stable under $s_{\beta'}$. Furthermore if $s_{\beta'}|_{\Delta(\beta)}$ is not the identity, then $s_{\beta'} \in W(\beta)$.*

Proof. First assume there exists a chain cascade C containing both β and β' . Without loss assume C is given by (1.2) and $\beta = \beta_i$ and $\beta' = \beta_j$ for some $i, j \leq k$. Then $\Delta(\beta)$ is clearly element-wise fixed by $s_{\beta'}$ if $j < i$ and $s_{\beta'} \in W(\beta)$ if $j \geq i$. Thus it remains to consider only the case when $C(\beta)$ is not a subchain of $C(\beta')$ and vice versa. But then the result follows from Proposition 1.7. QED

Since the elements in \mathcal{B} are orthogonal to one another the reflections s_β evidently commute with one another. The long element w_o of the Weyl group W is given in terms of the product of these commuting reflections. Let $\Delta_- = -\Delta_+$ and $\Delta(\varphi)_- = -\Delta(\varphi)_+$ for any $\varphi \in \Delta_+$.

Proposition 1.10. *One has*

$$w_o = \prod_{\beta \in \mathcal{B}} s_\beta \tag{1.7}$$

noting that the order of the product is immaterial because of commutativity. Furthermore w_o stabilizes $\mathfrak{g}(\beta)$ for any $\beta \in \mathcal{B}$ and

$$w_o|_{\mathfrak{g}(\beta)} = w_o(\beta)|_{\mathfrak{g}(\beta)}. \tag{1.8}$$

Proof. Let $\kappa \in W$ be given by the right side of (1.7). Clearly

$$\kappa(\Delta(\beta)) = \Delta(\beta) \tag{1.9}$$

for any $\beta \in \mathcal{B}$ by Proposition 1.9. But also clearly

$$\kappa(\beta) = -\beta \tag{1.10}$$

for any $\beta \in \mathcal{B}$ so that κ carries the highest root of $\mathfrak{g}(\beta)$ to the lowest root of $\mathfrak{g}(\beta)$. But $E(\beta) \subset \Delta(\beta)_+$. Hence $\kappa(E(\beta)) \subset \Delta(\beta)_-$. In particular

$$\kappa(E(\beta)) \subset \Delta_- \tag{1.11}$$

for any $\beta \in \mathcal{B}$. But then $\kappa(\Delta_+) = \Delta_-$ by (1.5). Thus $\kappa = w_o$. But then one has (1.8) by (1.9). QED

2. The coadjoint action

2.1. Let G be a simply connected Lie group where $\mathfrak{g} = \text{Lie } G$ and let $N \subset G$ be the subgroup corresponding to \mathfrak{n} . Let \mathfrak{n}_- be the span of all $e_{-\varphi}$ for $\varphi \in \Delta_+$. One has the direct sum

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}. \quad (2.1)$$

Let $\Phi : \mathfrak{g} \rightarrow \mathfrak{n}_-$ be the projection with kernel \mathfrak{b} . Using the Killing form (x, y) we identify the dual space \mathfrak{n}^* to \mathfrak{n} with \mathfrak{n}_- . Let $v \in \mathfrak{n}$, $u \in N$ and $z \in \mathfrak{n}_-$. The coadjoint action of v , $\text{coad } v$, and u , $\text{Coad } u$, on \mathfrak{n}_- is given by

$$\begin{aligned} \text{coad } v(z) &= \Phi[v, z] \\ \text{Coad } u(z) &= \Phi \text{Ad } u(z). \end{aligned} \quad (2.2)$$

Let $m = \text{card } \mathcal{B}$ and let \mathfrak{r} be the commutative m -dimensional subalgebra of \mathfrak{n} spanned by e_β for $\beta \in \mathcal{B}$. Let $R \subset N$ be the commutative unipotent subgroup corresponding to \mathfrak{r} . Let $\mathfrak{r}_- \subset \mathfrak{n}_-$ be the span of $e_{-\beta}$ for $\beta \in \mathcal{B}$. For any $z \in \mathfrak{r}_-$, $\beta \in \mathcal{B}$, let $a_\beta(z) \in \mathbb{C}$ be defined so that

$$z = \sum_{\beta \in \mathcal{B}} a_\beta(z) e_{-\beta} \quad (2.3)$$

and let

$$\mathfrak{r}_-^\times = \{\tau \in \mathfrak{r}_- \mid a_\beta(\tau) \neq 0, \forall \beta \in \mathcal{B}\}. \quad (2.4)$$

As an algebraic subvariety of \mathfrak{n}_- clearly

$$\mathfrak{r}_-^\times \cong (C^\times)^m. \quad (2.5)$$

For any $z \in \mathfrak{n}_-$ let O_z be the N -coadjoint orbit containing z . Let $N_z \subset N$ be the coadjoint isotropy subgroup at z and let $\mathfrak{n}_z = \text{Lie } N_z$. Since the action is algebraic N_z is connected and hence as N -spaces

$$O_z \cong N/N_z. \quad (2.6)$$

Let $\mathfrak{s} \subset \mathfrak{n}$ be the span of e_φ for $\varphi \in \Delta_+/\mathcal{B}$ so that one has the vector space direct sum

$$\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{s} \quad (2.7)$$

and let \mathfrak{s}_- be the span of $e_{-\varphi}$ for $\varphi \in \Delta_+/\mathcal{B}$ so that one has the vector space direct sum

$$\mathfrak{n}_- = \mathfrak{r}_- \oplus \mathfrak{s}_-. \quad (2.8)$$

Also for any $\beta \in \mathcal{B}$ let $\mathfrak{s}(\beta)$ be the span of e_φ for $\varphi \in E(\beta)/\{\beta\}$ so that the Heisenberg

$$\mathfrak{e}(\beta) = \mathfrak{s}(\beta) \oplus \mathbb{C} e_\beta. \quad (2.9)$$

Let $\mathfrak{s}(-\beta)$ be the span of $e_{-\varphi}$ for $\varphi \in E(\beta)/\{\beta\}$. One has the direct sums

$$\mathfrak{s} = \bigoplus_{\beta \in \mathcal{B}} \mathfrak{s}(\beta) \quad (2.10)$$

and

$$\mathfrak{s}_- = \bigoplus_{\beta \in \mathcal{B}} \mathfrak{s}(-\beta). \quad (2.11)$$

Let $\beta \in \mathcal{B}$. Since $\mathfrak{e}(\beta)$ is a Heisenberg Lie algebra, given $\varphi \in E(\beta)/\{\beta\}$, there exists a unique $\gamma \in E(\beta)/\{\beta\}$ such that $\varphi + \gamma = \beta$. We refer to γ as the Heisenberg twin to φ and the pair $\{\varphi, \gamma\}$ as Heisenberg twins. The following lemmas lead to a considerable simplification in dealing with the coadjoint action of N on \mathfrak{n}_- .

Lemma 2.1.. *Let $\beta \in \mathcal{B}$. Assume $\varphi, \varphi' \in \Delta_+$ and*

$$\beta = \varphi + \varphi'. \quad (2.12)$$

Then φ and φ' are both in $E(\beta)/\{\beta\}$ and are Heisenberg twins.

Proof. The equality (2.12) immediately implies that both φ and φ' are in $\Delta(\beta)$. But since $(\beta, \beta) \neq 0$ the equality (2.12) also implies that φ and φ' cannot

both be orthogonal to β . Hence at least one of the two must be in $E(\beta)/\{\beta\}$. But then the other is also in $E(\beta)/\{\beta\}$ by the existence of a Heisenberg twin. QED

Lemma 2.2. *Let $\beta, \beta' \in \mathcal{B}$ and let $x \in \mathfrak{s}(\beta)$. Assume $x \neq 0$. Then if $\beta' \neq \beta$, one has*

$$\text{coad } x(e_{-\beta'}) = 0. \quad (2.13)$$

On the other hand if $y = \text{coad } x(e_{-\beta})$, then

$$y \neq 0 \quad (2.14)$$

and

$$y \in \mathfrak{s}(-\beta). \quad (2.15)$$

Proof. Let $\varphi \in E(\beta)/\{\beta\}$ and let

$$z = \text{coad } e_\varphi(e_{-\beta'}). \quad (2.16)$$

Assume $z \neq 0$. Then clearly there exists $\varphi' \in \Delta_+$ such that $z = c e_{-\varphi'}$ for some nonzero scalar c . But then $(e_{-\beta'}, [e_\varphi, e_{\varphi'}]) \neq 0$. Hence

$$\beta' = \varphi + \varphi'. \quad (2.17)$$

But then φ and φ' are Heisenberg twins in $E(\beta')/\{\beta'\}$ by Lemma 2.1. Thus $z \neq 0$ implies $\beta = \beta'$. This proves (2.13) by writing x as a sum of root vectors for roots in $E(\beta)/\{\beta\}$.

Now assume $\beta = \beta'$. Let φ' be the Heisenberg twin to φ in $E(\beta)/\{\beta\}$. But then clearly $z = c e_{-\varphi'}$ for some nonzero scalar c . But then (2.14) and (2.15) follow immediately.

QED

Theorem 2.3. *Let $\tau \in \mathfrak{r}_-^\times$. Then (independent of τ) $N_\tau = R$ so that (2.6) becomes*

$$O_\tau \cong N/R. \quad (2.18)$$

Proof. Let $\beta \in \mathcal{B}$. Then, by strong orthogonality,

$$[e_\beta, \mathfrak{r}_-] = \mathbb{C} h_\beta \subset \mathfrak{h} \quad (2.19)$$

so that $\text{coad } e_\beta(\mathfrak{r}_-) = 0$. Hence

$$\mathfrak{r} \subset \mathfrak{n}_\tau. \quad (2.20)$$

Conversely let $v \in \mathfrak{n}_\tau$. We must show that $v \in \mathfrak{r}$. Assume not. Then we may assume that $v \in \mathfrak{s}$ and $v \neq 0$. Now by (2.10) we may write $v = \sum_{\beta \in \mathcal{B}} v(\beta)$ where $v(\beta) \in \mathfrak{s}(\beta)$. Let $\mathcal{B}_v = \{\beta \in \mathcal{B} \mid v(\beta) \neq 0\}$. Then \mathcal{B}_v is not empty. But by Lemma 2.2, $\text{coad } v(\beta)(\tau) \neq 0$ and $\text{coad } v(\beta)(\tau) \in \mathfrak{s}_{-\beta}$ for $\beta \in \mathcal{B}_v$. Hence $\text{coad } v(\tau) \neq 0$ by (2.11). This is a contradiction. QED

Remark 2.4. In effect Theorem 2.3. depends on the fact, established in the proof, that if $0 \neq v \in \mathfrak{s}$ and $\tau \in \mathfrak{r}_-^\times$, then

$$0 \neq \text{coad } v(\tau) \in \mathfrak{s}_-. \quad (2.21)$$

Theorem 2.5. *If $\tau, \tau' \in \mathfrak{r}_-^\times$ are distinct, then $O_\tau \cap O_{\tau'} = \emptyset$ so that one has a disjoint union*

$$\text{Coad } N(\mathfrak{r}_-^\times) = \bigcup_{\tau \in \mathfrak{r}_-^\times} O_\tau. \quad (2.22)$$

Proof. It suffices to show that

$$O_\tau \cap \mathfrak{r}_-^\times = \{\tau\}. \quad (2.23)$$

Let $u \in N$. We may write $u = \exp z$ where $z \in \mathfrak{n}$. If $z \in \mathfrak{r} = \mathfrak{n}_\tau$, then $\text{Coad } u(\tau) = \tau$. Hence we may assume that $z \notin \mathfrak{r}$ so that we can write $z = x + v$ where $x \in \mathfrak{r}$ and $0 \neq v \in \mathfrak{s}$. By Remark 2.3, if $y = \text{coad } z(\tau)$, then $0 \neq y \in \mathfrak{s}_-$. Thus we may write

$$y = \sum_{\varphi \in \Delta_+ \setminus \mathcal{B}} c_\varphi e_{-\varphi}. \quad (2.24)$$

Let $\varphi' \in \Delta_+ \setminus \mathcal{B}$ be a maximum element, relative to the ordering (1.1), such that $c_{\varphi'} \neq 0$. But then it is clear, from the exponentiation, that if we write

$$\text{Coad } u(\tau) - \tau = \sum_{\varphi \in \Delta_+} d_{\varphi} e_{-\varphi} \quad (2.25)$$

one has $d(\varphi') = c(\varphi')$. Hence $\text{Coad } u(\tau) \notin \mathfrak{r}_-$. In particular $\text{Coad } u(\tau) \notin T$. QED

2.2. Let $H \subset G$ be the subgroup corresponding to \mathfrak{h} so that $B = NH$ is a Borel subgroup of G and $\mathfrak{b} = \text{Lie } B$. Since B normalizes N the dual space \mathfrak{n}_- to \mathfrak{n} is a B -module where H operates via the adjoint representation and of course N operates via the coadjoint representation. Obviously the decompositions (2.7) and (2.8) are preserved by the action of H . Furthermore from the linear independence of the elements of \mathcal{B} one notes

Remark 2.6. The Zariski open subvariety $\mathfrak{r}_-^{\times} \subset \mathfrak{r}_-$ is stable under the action of H and in fact H operates transitively on \mathfrak{r}_-^{\times} so that \mathfrak{r}_-^{\times} is isomorphic to a homogeneous space for H .

The action of H on \mathfrak{r}_-^{\times} extends to an action of H on the corresponding set $\{O_{\tau}, \tau \in \mathfrak{r}_-^{\times}\}$ of N -coadjoint orbits. Since H normalizes N the following statement is obvious.

Proposition 2.7. *For any $\tau \in \mathfrak{r}_-^{\times}$ and $a \in H$ one has*

$$O_{\text{Ad } a(\tau)} = \text{Ad } a(O_{\tau}). \quad (2.26)$$

Let

$$X = \cup_{\tau \in \mathfrak{r}_-^{\times}} O_{\tau} \quad (2.27)$$

so that, by Theorem 2.5, the union (2.27) is disjoint.

Clearly X (see Remark 2.6) is an orbit of the action of B on \mathfrak{n}_- so that X has the structure of an algebraic subvariety of \mathfrak{n}_- which is Zariski open in its closure.

On the other hand the product variety $N/R \times \mathfrak{r}_-^\times$ is an affine variety and one has a bijection

$$\psi : N/R \times \mathfrak{r}_-^\times \rightarrow X, \quad (2.28)$$

where if $[u] \in N/R$ denotes the left coset of $u \in N$ in N/R , one has

$$\psi([u], \tau) = \text{Coad } u(\tau). \quad (2.29)$$

But since H normalizes both N and R it follows easily that N/R is a B -homogeneous space. The action of H on \mathfrak{r}_-^\times extends to an action of B on \mathfrak{r}_-^\times where N operates trivially. Consequently $N/R \times \mathfrak{r}_-^\times$ has the structure of a B -homogeneous space.

Theorem 2.8. *The map ψ (see (2.28) and (2.29)) is a B -isomorphism of affine B -homogeneous spaces. Furthermore X is Zariski open in \mathfrak{n}_- so that*

$$\overline{X} = \mathfrak{n}_-. \quad (2.30)$$

Proof. We have noted that $N/R \times \mathfrak{r}_-^\times$ is an affine variety. Since a homogeneous space of an affine algebraic group inherits a unique algebraic structure, to establish the first statement, it suffices only to see that ψ is a B -map. But this is immediate. But $\dim \mathfrak{r}_-^\times = \dim R$. Thus $\dim X = \dim \mathfrak{n}_-$. This proves (2.30). QED

3. The characters of H on $S(\mathfrak{n})^N$

3.1. Let $\Lambda \subset \mathfrak{h}^*$ be the weight lattice for $(\mathfrak{h}, \mathfrak{g})$ and let Λ_{ad} be the root sublattice of Λ . For each $\nu \in \Lambda_{ad}$ let χ_ν be the character on H defined so that for $a = \exp x$, $x \in \mathfrak{h}$, one has $\chi_\nu(a) = e^{\nu(x)}$. Of course the character group, \widehat{H} , of H is given by

$$\widehat{H} = \{\chi_\nu \mid \nu \in \Lambda_{ad}\}.$$

Recalling that $m = \text{card } \mathcal{B}$ let $\Lambda(\mathcal{B}) \subset \Lambda_{ad}$ be the free abelian group of rank m , generated by \mathcal{B} , and let

$$\widehat{H}(\mathcal{B}) = \{\chi_\nu \mid \nu \in \Lambda(\mathcal{B})\}.$$

If V is a affine variety (over \mathbb{C}) we will let $A(V)$ denote the affine algebra of regular functions on V . The quotient field of $A(V)$, the algebra of rational functions on V , will be denoted by $Q(V)$. If a linear algebraic group G' operates algebraically on V , then G' operates as a group of automorphisms of $A(V)$ so that if $g \in G'$, $\phi \in A(V)$ and $v \in V$, then $g \cdot \phi(v) = \phi(g^{-1} \cdot v)$. The group also operates as a group of automorphisms of $Q(V)$ where, for $g \in G'$, $\phi, \phi' \in A(V)$ and $\phi' \neq 0$, then $g \cdot \phi/\phi' = g \cdot \phi/g \cdot \phi'$. If $\mathfrak{g}' = \text{Lie } G'$, then \mathfrak{g}' operates as a Lie algebra of derivations of $A(V)$ and $Q(V)$. Using the fact that, as one knows, $G' \cdot \phi$ spans a finite-dimensional subspace of $A(V)$ for any $\phi \in A(V)$, one has $x \cdot \phi = \frac{d}{dt}(\exp t x \cdot \phi)|_{t=0}$. If $0 \neq \phi' \in A(V)$, then $x \cdot \phi/\phi' = (\phi' x \cdot \phi - \phi x \cdot \phi')/(\phi')^2$. If M is any G' module, then $M^{G'}$ will be the submodule of G' invariants in M .

Now the map ψ (see (2.28)) induces a B -isomorphism

$$A(X) \rightarrow A(N/R) \otimes A(\mathfrak{r}_-^\times) \quad (3.1)$$

by Theorem 2.8. But then, noting the action of N , the map (3.1) defines an H -isomorphism

$$A(X)^N \rightarrow A(\mathfrak{r}_-^\times). \quad (3.2)$$

But then, recalling the affine algebra of a (complex) torus one immediately has

Theorem 3.1. *For any $\nu \in \Lambda(\mathcal{B})$ there exists a unique (up to scalar multiplication) nonzero H -weight vector ξ_ν in $A(X)^N$ of weight ν . That is, for any $a \in H$,*

$$a \cdot \xi_\nu = \chi_\nu(a) \xi_\nu. \quad (3.3)$$

Moreover the set, $\{\xi_\nu \mid \nu \in \Lambda(\mathcal{B})\}$, of weight vectors are a basis of $A(X)^N$. In particular every weight in $A(X)^N$ occurs with multiplicity one and only weights in $\Lambda(\mathcal{B})$ occur.

3.2. The Killing form pairing of \mathfrak{n} and \mathfrak{n}_- identifies the symmetric algebra

$S(\mathfrak{n})$ with $A(\mathfrak{n}_-)$ and the quotient field $F(\mathfrak{n})$ of $S(\mathfrak{n})$ with $Q(\mathfrak{n}_-)$. Of course $S(\mathfrak{n})$ is a unique factorization domain.

Remark 3.2. It is evident that if $\phi \in S(\mathfrak{n})$ is a prime polynomial and $b \in B$, then $b \cdot \phi$ is again a prime polynomial and $\psi = \phi_1 \cdots \phi_k$ is the prime factorization of $0 \neq \psi \in S(\mathfrak{n})$, then

$$b \cdot \psi = (b \cdot \phi_1) \cdots (b \cdot \phi_k) \quad (3.4)$$

is the prime factorization of $b \cdot \psi$.

Proposition 3.3. $Q(\mathfrak{n}_-)^N$ is the quotient field of $S(\mathfrak{n})^N$. Furthermore the prime factors of any $0 \neq \psi \in S(\mathfrak{n})^N$ are also in $S(\mathfrak{n})^N$.

Proof. Let $0 \neq \gamma \in Q(\mathfrak{n}_-)^N$. Write $\gamma = \phi/\psi$ where $\phi, \psi \in S(\mathfrak{n})$. Then $\psi\gamma = \phi$ and hence for any $u \in N$ one has

$$(u \cdot \psi)\gamma = u \cdot \phi. \quad (3.5)$$

But $N \cdot \psi$ spans a finite-dimensional N -submodule M of $S(\mathfrak{n})$. By the unipotence of N and its action on M there exists $0 \neq \psi' \in M^N \subset S(\mathfrak{n})^N$. But then if $\phi' = \psi'\gamma$, it follows from (3.5) that $\phi' \in S(\mathfrak{n})^N$. But $\gamma = \phi'/\psi'$. This proves the first statement of the proposition.

Now let $0 \neq \psi \in S(\mathfrak{n})^N$ and let $\psi = \phi_1 \cdots \phi_k$ be a prime factor decomposition of ψ . But then for any $u \in N$,

$$\psi = (u \cdot \phi_1) \cdots (u \cdot \phi_k) \quad (3.6)$$

is another prime factor decomposition of ψ . By the continuity of the action of N and the uniqueness (up to scalar multiplication) of the prime factor decomposition, for any $j \in \{1, \dots, k\}$, there exists $c \in \mathbb{C}^\times$ such that $u \cdot \phi_j = c\phi_j$. But, by unipotence, 1 is the only eigenvalue of the action of u on $S(\mathfrak{n})$. Thus $c = 1$ and hence $\phi_j \in S(\mathfrak{n})^N$. QED

Now, by (2.30), one has $Q(X) = Q(\mathfrak{n}_-)$. Thus we have the B -inclusions

$$S(\mathfrak{n}) \subset A(X) \subset Q(\mathfrak{n}_-). \quad (3.7)$$

Remark 3.4. Note that in (3.7), as functions on \mathfrak{n}_- , the elements of $S(\mathfrak{n})$ are exactly the functions in $A(X)$ which extend, as regular functions, (i.e., everywhere defined rational functions) to all of \mathfrak{n}_- .

But now (3.7) yields the inclusion

$$S(\mathfrak{n})^N \subset A(X)^N \quad (3.8)$$

of (completely reducible) H -modules. Recalling Theorem 3.1 let

$$\Lambda_{\mathfrak{n}}(\mathcal{B}) = \{\nu \in \Lambda(\mathcal{B}) \mid \xi_{\nu} \in S(\mathfrak{n})^N\}. \quad (3.9)$$

Also let $\Lambda_{\text{dom}}(\mathcal{B})$ be the set of all dominant weights in $\Lambda(\mathcal{B})$. In the following result it is proved that $\Lambda_{\mathfrak{n}}(\mathcal{B}) \subset \Lambda_{\text{dom}}(\mathcal{B})$. It is established as argued in the Introduction that in fact $\Lambda_{\mathfrak{n}}(\mathcal{B}) = \Lambda_{\text{dom}}(\mathcal{B})$.

Theorem 3.5. *Every H -weight in $S(\mathfrak{n})^N$ occurs with multiplicity 1. Moreover $\Lambda_{\mathfrak{n}}(\mathcal{B})$ is the set of such H -weights. Furthermore*

$$\{\xi_{\nu} \mid \nu \in \Lambda_{\mathfrak{n}}(\mathcal{B})\} \quad (3.10)$$

is a basis of $S(\mathfrak{n})^N$. Finally

$$\Lambda_{\mathfrak{n}}(\mathcal{B}) \subset \Lambda_{\text{dom}}(\mathcal{B}). \quad (3.11)$$

Proof. Except for (3.10) the theorem follows immediately from (3.8) and Theorem 3.1. The inclusion (3.11) follows immediately from the fact that ξ_{ν} for $\nu \in \Lambda_{\mathfrak{n}}(\mathcal{B})$ is necessarily the highest weight vector for the \mathfrak{g} -submodule, in the symmetric algebra $S(\mathfrak{g})$, generated by ξ_{ν} . QED

Let

$$\mathcal{P} = \{\nu \in \Lambda_{\mathfrak{n}}(\mathcal{B}) \mid \xi_{\nu} \text{ is a prime polynomial in } S(\mathfrak{n})\}. \quad (3.12)$$

Theorem 3.6. *One has $\text{card } \mathcal{P} = m$ where, we recall $m = \text{card } \mathcal{B}$, so that we can write*

$$\mathcal{P} = \{\mu_1, \dots, \mu_m\}. \quad (3.13)$$

Furthermore the weights μ_i in \mathcal{P} are linearly independent and the set P of prime polynomials, ξ_{μ_i} , $i = 1, \dots, m$, are algebraically independent. In addition one has a bijection

$$\Lambda_{\mathfrak{n}}(\mathcal{B}) \rightarrow (\mathbb{N})^m, \quad \nu \mapsto (d_1(\nu), \dots, d_m(\nu)) \quad (3.14)$$

such that, writing $d_i = d_i(\nu)$, up to scalar multiplication,

$$\xi_{\nu} = \xi_{\mu_1}^{d_1} \cdots \xi_{\mu_m}^{d_m} \quad (3.15)$$

and (3.15) is the prime factorization of ξ_{ν} for any $\nu \in \Lambda_{\mathfrak{n}}\mathcal{B}$. Finally

$$S(\mathfrak{n})^N = \mathbb{C}[\xi_{\mu_1}, \dots, \xi_{\mu_m}] \quad (3.16)$$

so that $S(\mathfrak{n})^N$ is a polynomial ring in m generators.

Proof. Let $\nu \in \Lambda_{\mathfrak{n}}(\mathcal{B})$ and consider the prime factorization of ξ_{ν} . We use the notation of Remark 3.2 where $\psi = \xi_{\nu}$ and $b \in H$. Then since $b \cdot \xi_{\nu} = \chi_{\nu}(b) \xi_{\nu}$ it follows from (3.4) that the right side of (3.4) is another prime factorization of ξ_{ν} . By the continuity of the action of H and the uniqueness of the factorization it follows that for any $j = 1, \dots, k$, there exists $\chi_j \in \widehat{H}$ such that $b \cdot \phi_j = \chi_j(b) \phi_j$ for all $b \in H$. But then, by Theorem 3.1 and Theorem 3.5, one has $\chi_j = \chi_{\nu_j}$ for a unique $\nu_j \in \Lambda_{\mathfrak{n}}(\mathcal{B})$ and up to scalar multiplication $\phi_j = \xi_{\nu_j}$. But also $\nu_j \in \mathcal{P}$. Thus for one thing this shows \mathcal{P} is not empty and up to scalar multiplication

$$\xi_{\nu} = \xi_{\nu_1} \cdots \xi_{\nu_k} \quad (3.17)$$

and (3.17) is the prime factorization of ξ_ν .

For a positive integer $r \leq \text{card } \mathcal{P}$ let μ'_1, \dots, μ'_r be r distinct elements of \mathcal{P} . For any $d = (d_1, \dots, d_r) \in (\mathbb{Z}_+)^r$ let $\nu(d) \in \Lambda(\mathcal{B})$ be defined by putting $\nu(d) = \sum_{j=1}^r d_j \mu'_j$. But, up to scalar multiplication,

$$\xi_{\nu(d)} = \xi_{\mu'_1}^{d_1} \cdots \xi_{\mu'_r}^{d_r} \quad (3.18)$$

so that $\nu(d) \in \Lambda_{\mathbf{n}}(\mathcal{B})$ and (3.18) is the prime factorization of $\xi_{\nu(d)}$. One also has a map

$$(\mathbb{Z}_+)^r \rightarrow \Lambda_{\mathbf{n}}(\mathcal{B}), \quad d \mapsto \nu(d). \quad (3.19)$$

But then, by the uniqueness of the prime factorization, the map (3.19) is necessarily injective. But weight vectors belonging to distinct weights are linearly independent (see also Theorem 3.5) so that no nontrivial linear combination of the monomials on the right side of (3.18) can vanish. This proves

$$\xi_{\mu'_1}, \dots, \xi_{\mu'_r} \text{ are algebraically independent.} \quad (3.20)$$

But by (2.5) and (3.2) the transcendence degree of $A(X)^N$ is m . But

$$S(\mathbf{n})^N \subset A(X)^N \subset Q(\mathbf{n}_-)^N \quad (3.21)$$

by (3.6). Hence

$$\text{the transcendence degree of } S(\mathbf{n})^N \text{ is } m \quad (3.22)$$

by Proposition 3.2. Thus $r \leq m$ and hence if $n = \text{card } \mathcal{P}$ one has $n \leq m$. But then if we choose $r = n$, the map (3.19) is surjective by (3.17) and one must have $S(\mathbf{n})^N = \mathbb{C}[\xi_{\mu'_1}, \dots, \xi_{\mu'_n}]$. But then $n = m$ by (3.21). Except for the statement that ν_i are linearly independent weights this proves Theorem 3.6, noting that (3.14) is the inverse of the bijection (3.19) when we choose $\mu_j = \mu'_j$. Assume that the μ_i are not linearly dependent. Since these weights lie in a lattice there exists, over the rational

numbers, a vanishing nontrivial linear combination. Clearing denominators such a linear combination exists over the integers. But this implies that there exists distinct $d, d' \in (\mathbb{N})^m$ such that $\nu(d) = \nu(d')$. This contradicts the bijectivity of (3.14). QED

Remark 3.7. We note the following uniqueness statement. If $\{\nu_1, \dots, \nu_k\}$ is any subset of $\Lambda_n(\mathcal{B})$ with the property that there exists a bijection

$$\Lambda_n(\mathcal{B}) \rightarrow (\mathbb{Z}_+)^k, \quad \nu \mapsto (e_1(\nu), \dots, e_k(\nu))$$

such that, writing $e_i = e_i(\nu)$,

$$\xi_\nu = \xi_{\nu_1}^{e_1} \cdots \xi_{\nu_k}^{e_k}$$

up to scalar multiplication, then one must have $k = m$ and $\{\nu_1, \dots, \nu_k\}$ is some reordering of $\mathcal{P} = \{\mu_1, \dots, \mu_m\}$. This is clear since otherwise one would have a contradiction of the primeness of the ξ_{μ_j} . In particular the generators constructed by A. Joseph in [J] of (without any reference to primeness) must necessarily be the same as our ξ_{ν_j} .

As to all the weights ν in the group (not semigroup) $\Lambda(\mathcal{B})$ (see Theorem 3.1), one has an extension of the map (3.14) involving \mathbb{Z} (and hence negative integers as well).

Theorem 3.8. *There exists a bijection*

$$\Lambda(\mathcal{B}) \rightarrow (\mathbb{Z})^m, \quad \nu \mapsto (d_1(\nu), \dots, d_m(\nu)) \tag{3.23}$$

such that, writing $d_i = d_i(\nu)$,

$$\xi_\nu = \xi_{\mu_1}^{d_1} \cdots \xi_{\mu_m}^{d_m} \tag{3.24}$$

up to scalar multiplication, recalling that the ξ_ν , for $\nu \in \Lambda(\mathcal{B})$ is a basis of $A(X)^N$. See Theorem 3.1.

Proof. Let $\nu \in \Lambda(\mathcal{B})$. By (3.21) and Proposition 3.3, up to scalar multiplication, one can uniquely write $\xi_\nu = p/q$ where $p, q \in S(\mathfrak{n})^N$ and p and q are prime to one another. But then if $a \in H$ one has $\chi_\nu(a) = a \cdot p/a \cdot q$. By uniqueness both p and q must be weight vectors in $S(\mathfrak{n})^N$. But then Theorem 3.8 follows from Theorem 3.6. QED

Remark 3.9. It follows from Theorem 3.8 that both \mathcal{B} and \mathcal{P} are bases for the free abelian group $\Lambda(\mathcal{B})$. Consequently there must be a matrix in $Sl(m, \mathbb{Z})$ which expresses one such basis in terms of the other.

For $\nu \in \mathfrak{h}^*$ and $\beta \in \mathcal{B}$ let $r_\beta(\nu) = (\beta, \nu)/(\beta, \beta)$ so that, by (1.7),

$$\begin{aligned} w_o(\nu) = -\nu &\iff \nu \text{ is in the span of } \mathcal{B} \\ &\iff \nu = \sum_{\beta \in \mathcal{B}} r_\beta(\nu) \beta. \end{aligned} \tag{3.25}$$

In any case put

$$r(\nu) = \sum_{\beta \in \mathcal{B}} r_\beta(\nu). \tag{3.26}$$

Let Λ_{dom} be the set of all dominant weights so that

Remark 3.10. If $\nu \in \Lambda_{dom}$ note that, for all $\beta \in \mathcal{B}$, $r_\beta(\nu) \in \mathbb{Z}_+/2$ and if $\nu \in \Lambda_{dom}(\mathcal{B})$ then $w_o\nu = -\nu$ and $r_\beta(\nu) \in \mathbb{Z}_+$. In particular this is true for $\nu \in \Lambda_n(\mathcal{B})$ by (3.11).

3.3. Let Γ be the set of all maps $\gamma : \Delta_+ \rightarrow \mathbb{Z}_+$. For $\gamma \in \Gamma$ let $d(\gamma) = \sum_{\varphi \in \Delta_+} \gamma(\varphi)$ and let γ^{root} be the element in the root lattice given by putting $\gamma^{\text{root}} = \sum_{\varphi \in \Delta_+} \gamma(\varphi) \varphi$. Also let $z_\gamma \in S^{d(\gamma)}(\mathfrak{n})$ be given by putting

$$z_\gamma = \prod_{\varphi \in \Delta_+} e_\varphi^{\gamma(\varphi)}. \tag{3.27}$$

Let $\nu \in \Lambda_n(\mathcal{B})$. Now from the multiplicity 1 condition (see Theorem 3.1) it follows that

$$\xi_\nu, \text{ for any } \nu \in \Lambda_n(\mathcal{B}), \text{ is a homogeneous polynomial in } S(\mathfrak{n}). \tag{3.28}$$

Let $\deg \nu$ be the degree of the homogeneous polynomial ξ_ν and let

$$\Gamma(\nu) = \{\gamma \in \Gamma \mid d(\gamma) = \deg \nu \text{ and } \gamma^{\text{root}} = \nu\}.$$

But then there exists scalars s_γ , $\gamma \in \Gamma(\nu)$ such that

$$\xi_\nu = \sum_{\gamma \in \Gamma(\nu)} s_\gamma z_\gamma. \quad (3.29)$$

But now since X is Zariski open in \mathfrak{n}_- (see Theorem 2.8) there exists $x \in X$ such that $\xi_\nu(x) \neq 0$. But $x = \text{Coad } b^{-1}(t)$ for $b \in N$ and $t \in \mathfrak{r}_-^\times$. Write

$$t = \sum_{\beta \in \mathcal{B}} t_\beta e_{-\beta}, \quad (3.30)$$

where $t_\beta \in \mathbb{C}^\times$ for all $\beta \in \mathcal{B}$. But by N -invariance

$$\begin{aligned} 0 &\neq \xi_\nu(x) \\ &= \xi_\nu(t). \end{aligned} \quad (3.31)$$

But $z_\gamma(t) = 0$ for $\gamma \in \Gamma(\nu)$ unless $\gamma = \gamma_{\mathfrak{r}}$ where $\gamma_{\mathfrak{r}}(\varphi) = 0$ for $\varphi \notin \mathcal{B}$. But any such $\gamma_{\mathfrak{r}}$ is clearly unique where necessarily $\gamma(\beta) = r_\beta(\nu)$, for all $\beta \in \mathcal{B}$. Hence, also one must have $\deg \nu = r(\nu)$. We have in fact thus proved

Theorem 3.11. *Let $\nu \in \Lambda_{\mathfrak{n}}(\mathcal{B})$. Then $\deg \nu = r(\nu)$. Furthermore (see (3.29)) one must have $s_{\gamma_{\mathfrak{r}}} \neq 0$ and if $t \in \mathfrak{r}_-^\times$ is given by (3.30), then*

$$\xi_\nu(t) = s_{\gamma_{\mathfrak{r}}} \prod_{\beta \in \mathcal{B}} t_\beta^{r_\beta(\nu)}. \quad (3.32)$$

References

- [D] Jacques Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotent*.
IV, Canad. J. Math., **11**(1959), 321–344.

- [LW] Ronald Lipsman and Joseph Wolf, *Canonical semi-invariants and the Plancherel formula for parabolic groups*, Trans. Amer. Math. Soc., **269**(1982), 111–131.
- [J] Anthony Joseph, *A preparation theorem for the prime spectrum of a semisimple Lie algebra*, Journ. of Alg., **48**, No.2, (1977), 241–289.
- [K] Bertram Kostant, *Cent $U(\mathfrak{n})$ and a construction of Lipsman-Wolf*, arXiv /0178653, [math.RT], Jan 12, 2011.

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